

A Novel Symmetry in Sigma models

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Abstract

A class of non-linear sigma models possessing a new symmetry is identified. The same symmetry is also present in Chern-Simons theories. This hints at a possible topological origin for this class of sigma models. The non-linear sigma models obtained by non-Abelian duality are a particular case in this class.

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1. Introduction

Non-linear sigma models in two dimensions possess remarkable features due to their rich symmetries. The symmetry properties a sigma model can have depend very much on the form of the metric and the torsion tensors. The most studied non-linear sigma model is the Wess-Zumino-Novikov-Witten (WZNW) model. This model enjoys an extra symmetry, generating a current algebra [1], precisely when the metric and the torsion tensors take the very specific forms

$$\begin{aligned} G_{ij} &= e_i^a e_j^b \eta_{ab} = \tilde{e}_i^a \tilde{e}_j^b \eta_{ab} \\ H &= -\frac{1}{3} \text{tr} (e \wedge e \wedge e) = \frac{1}{3} \text{tr} (\tilde{e} \wedge \tilde{e} \wedge \tilde{e}) \quad , \end{aligned} \quad (1)$$

where we have introduced the differential forms on the group manifold [2]

$$\begin{aligned} e &= T_a e_i^a d\phi^i = g^{-1} dg \\ \tilde{e} &= T_a \tilde{e}_i^a d\phi^i = -dg g^{-1} \quad . \end{aligned} \quad (2)$$

The torsion tensor $H = \frac{1}{3!} H_{ijk} d\phi^i \wedge d\phi^j \wedge d\phi^k$ is related to the antisymmetric tensor field $B = \frac{1}{2!} B_{ij} d\phi^i \wedge d\phi^j$ by the usual relation $H = \frac{1}{2} dB$. Here η_{ab} is the invariant bi-linear form of the Lie algebra generated by T_a and g is a group element parametrised by ϕ^i .

In fact one can ask whether there exist other forms for G_{ij} and H_{ijk} which would lead to other forms of current algebras. This question was answered in refs.[3, 4, 5, 6, 7] where a generalisation of the above expressions for the metric and the torsion were found. The WZNW model is then just a particular case of this generalisation.

We explore, in this paper, the possibility of finding other non-linear sigma models that might have further symmetries depending on the forms of G_{ij} and B_{ij} . Indeed, we identify a class of sigma models which have a very interesting symmetry. Remarkably, the same symmetry appears in Chern-Simons theories in three dimensions. This hints at a possible connection between the two theories.

2. The new symmetry

Consider the action for a general bosonic two-dimensional non-linear sigma model

$$S(\varphi) = \int d^2x \sqrt{\gamma} \left(\gamma^{\mu\nu} G_{ij}(\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j + \tilde{\epsilon}^{\mu\nu} B_{ij}(\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j \right) \quad . \quad (3)$$

In this equation $\gamma_{\mu\nu}$ is the metric on the two-dimensional world sheet, γ is its determinant and $\hat{\epsilon}^{\mu\nu} = \frac{1}{\sqrt{\gamma}}\epsilon^{\mu\nu}$ is the alternating tensor. This action can be written as

$$S(\varphi) = \int d^2x \sqrt{\gamma} \left(\hat{\epsilon}^{\mu\nu} \eta_{ij} A_\mu^i \partial_\nu \varphi^j \right) , \quad (4)$$

where we have introduced the gauge field-like quantity A_μ^i

$$\begin{aligned} A_\mu^i &= R_{\mu\nu}^{ij} \eta_{jk} \hat{\epsilon}^{\nu\alpha} \partial_\alpha \varphi^k \\ R_{\mu\nu}^{ij} &= \eta^{ik} \eta^{jl} (\gamma_{\mu\nu} G_{kl} + \hat{\epsilon}_{\mu\nu} B_{kl}) \end{aligned} \quad (5)$$

with η_{ij} a symmetric field-independent metric whose inverse is η^{ij} . Suppose now that η_{ij} is the invariant bi-linear form of a Lie algebra whose structure constants we denote by f_{jk}^i (which means that $\eta_{ij} f_{kl}^j + \eta_{kj} f_{il}^j = 0$).

We would like to investigate under which conditions the action (4) has a symmetry of the form

$$\delta\varphi^i = f_{jk}^i \xi^j F_{\mu\nu}^k \hat{\epsilon}^{\mu\nu} , \quad (6)$$

where $\xi^j(x)$ is the infinitesimal gauge parameter and $F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + f_{jk}^i A_\mu^j A_\nu^k$ is the field strength of the gauge field A_μ^i as given by (5). The transformation is suggested by the form of the action (4). The same kind of symmetry was identified in the context of non-Abelian duality in sigma models [8] and in non-Abelian gauge theories [9]

We found that the action is invariant, up to a total derivative, provided that the metric G_{ij} and the antisymmetric tensor B_{ij} satisfy

$$\partial_k R_{\mu\nu}^{ij} = \eta_{kl} f_{mn}^l R_{\mu\alpha}^{im} R_{\nu\beta}^{jn} \hat{\epsilon}^{\alpha\beta} . \quad (7)$$

This condition can of course be expressed explicitly as two separate conditions on G_{ij} and B_{ij}

$$\begin{aligned} \partial_k G_{ij} &= f_{km}^l \eta^{mn} (G_{li} B_{nj} - G_{nj} B_{li}) \\ \partial_k B_{ij} &= -f_{km}^l \eta^{mn} (G_{li} G_{nj} + B_{li} B_{nj}) \end{aligned} \quad (8)$$

This shows the special geometry of this class of sigma models. In particular, the Riemann tensor and the torsion will be given in closed forms as products of G_{ij} , G^{ij} and B_{ij} .

Under these conditions, the equations of motion of the non-linear sigma model lead to

$$\hat{\epsilon}^{\mu\nu} F_{\mu\nu}^i = 0 . \quad (9)$$

Therefore the above transformation vanishes on-shell. As seen later, this equation can also be thought of as deriving from a Chern-Simons theory.

It is straightforward to find the unique solution to the symmetry invariance condition in (7). In order to do this, we denote by $\tilde{R}_{ij}^{\mu\nu}$ the inverse of $R_{\mu\nu}^{ij}$ (that is, $R_{\mu\nu}^{ij}\tilde{R}_{jk}^{\nu\alpha} = \delta_\mu^\alpha\delta_k^i$). Equation (7) is then cast into the first order differential equation

$$\partial_k \tilde{R}_{ij}^{\mu\nu} = -\eta_{kl} f_{ij}^l \tilde{\epsilon}^{\mu\nu} \quad (10)$$

whose general solution is given by

$$\tilde{R}_{ij}^{\mu\nu} = - \left[N_{ij}^{\mu\nu} + \tilde{\epsilon}^{\mu\nu} \eta_{kl} f_{ij}^l \varphi^k \right] , \quad (11)$$

where $N_{ij}^{\mu\nu} = N_{ji}^{\nu\mu}$ is any field-independent matrix, and in general $N_{\mu\nu}^{ij} = \gamma^{\mu\nu} A_{ij} + \tilde{\epsilon}^{\mu\nu} C_{ij}$.

The action can be cast into a form which is familiar in the context of non-Abelian duality [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. By extracting $\partial_\mu \varphi^i$ from (5) and eliminating it in (4), one finds, after some straightforward manipulations, the following action

$$S(\varphi, A) = \int d^2x \sqrt{\gamma} \left(N_{ij}^{\mu\nu} A_\mu^i A_\nu^j + \tilde{\epsilon}^{\mu\nu} \eta_{ij} F_{\mu\nu}^i \varphi^j \right) , \quad (12)$$

where we have ignored a total derivative. If one treats A_μ^i and φ^i as independent variables then the equations of motion for A_μ^i are precisely those in (5). Indeed, this action is obtained (when A_μ^i and φ^i are independent) by performing a non-Abelian duality transformation on the following action

$$S(g) = \int d^2x \sqrt{\gamma} N_{ij}^{\mu\nu} \eta^{ik} \eta^{jl} \text{tr} \left(T_k g^{-1} \partial_\mu g \right) \text{tr} \left(T_l g^{-1} \partial_\nu g \right) , \quad (13)$$

where T_i are the generators of the Lie algebra $[T_i, T_j] = f_{ij}^k T_k$, g is a Lie group element and tr is the invariant bi-linear form $\text{tr}(XY) = \eta_{ij} X^i Y^j$. This action is invariant under the global transformation $g \longrightarrow hg$. The non-Abelian dual theory is obtained by gauging this symmetry and at the same time restricting the gauge field strength to vanish [20]. We therefore obtain the action

$$S(g, \varphi, A) = \int d^2x \sqrt{\gamma} \left[N_{ij}^{\mu\nu} \eta^{ik} \eta^{jl} \text{tr} \left(T_k g^{-1} D_\mu g \right) \text{tr} \left(T_l g^{-1} D_\nu g \right) + \tilde{\epsilon}^{\mu\nu} \text{tr}(\varphi F_{\mu\nu}) \right] . \quad (14)$$

The covariant derivative is $D_\mu g = \partial_\mu g + A_\mu g$ with $A_\mu \longrightarrow h^{-1} A_\mu h - \partial_\mu h h^{-1}$, the gauge field is $A_\mu = T_i A_\mu^i$ and the Lagrange multiplier is $\varphi = T_i \varphi^i$ with $\varphi \longrightarrow h \varphi h^{-1}$. The gauge invariance allows us to choose a gauge such that $g = 1$. In this gauge $S(g, \varphi, A)$ reproduces precisely $S(g, A)$ as given by (12).

The dual of the principal chiral model is a special case of this construction [21, 22, 23]. The dual of the chiral model is obtained when $A_{ij} = \eta_{ij}$ and $C_{ij} = 0$.

3. Generalisation

Another important case is obtained by splitting the field φ^i of the previous non-linear sigma model as $\varphi^i = (X^a, Y^A)$ and restricting the transformation to the fields X^a only. In this case, the non-linear sigma model action takes the form

$$S(X, Y) = \int d^2x \sqrt{\gamma} \left(\hat{\epsilon}^{\mu\nu} \eta_{ab} A_\mu^a \partial_\nu X^b + \hat{\epsilon}^{\mu\nu} \hat{\epsilon}^{\alpha\beta} M_{\mu\alpha}^{aA} \eta_{ab} L_{AB} \partial_\beta Y^B \partial_\nu X^b + R_{AB}^{\mu\nu} \partial_\mu Y^A \partial_\nu Y^B \right) , \quad (15)$$

The gauge field A_μ^a involves both X^a and Y^A through

$$A_\mu^a = R_{\mu\nu}^{ab} \hat{\epsilon}^{\nu\alpha} \eta_{bc} \partial_\alpha X^c + R_{\mu\nu}^{aA} \hat{\epsilon}^{\nu\alpha} L_{AB} \partial_\alpha Y^B . \quad (16)$$

The different quantities introduced here can depend on both X^a and Y^A and are such that

$$\begin{aligned} R_{\mu\nu}^{ab} &= \eta^{ac} \eta^{bd} [\gamma_{\mu\nu} G_{cd} + \hat{\epsilon}_{\mu\nu} B_{cd}] \\ R_{AB}^{\mu\nu} &= [\gamma^{\mu\nu} G_{AB} + \hat{\epsilon}^{\mu\nu} B_{AB}] \\ R_{\mu\nu}^{aA} + M_{\mu\nu}^{aA} &= 2\eta^{ab} L^{AB} [\gamma_{\mu\nu} G_{bB} + \hat{\epsilon}_{\mu\nu} B_{bB}] . \end{aligned} \quad (17)$$

The symmetric matrices η_{ab} and L_{AB} are field-independent and their inverses are, respectively, η^{ab} and L^{AB} . We then suppose that η_{ab} is a bi-linear invariant form of a Lie algebra with structure constants f_{bc}^a .

Let us find the conditions under which the sigma model (15) is invariant under

$$\delta X^a = f_{bc}^a \xi^b F_{\mu\nu}^c \hat{\epsilon}^{\mu\nu} , \quad \delta Y^A = 0 . \quad (18)$$

We find that the action remains invariant, up to a total derivative, when the following conditions are fulfilled

$$\begin{aligned} M_{\mu\nu}^{aA} &= R_{\mu\nu}^{aA} \\ \partial_c R_{\mu\alpha}^{ad} &= \eta_{cb} f_{er}^b R_{\mu\sigma}^{ae} R_{\alpha\tau}^{dr} \hat{\epsilon}^{\sigma\tau} \\ \partial_c R_{\mu\alpha}^{aA} &= \eta_{cb} f_{er}^b R_{\mu\sigma}^{ae} R_{\alpha\tau}^{rA} \hat{\epsilon}^{\sigma\tau} \\ \partial_c R_{AB}^{\mu\alpha} &= \eta_{cb} f_{er}^b L_{AE} L_{BD} R_{\alpha\sigma}^{eE} R_{\tau\beta}^{rD} \hat{\epsilon}^{\sigma\mu} \hat{\epsilon}^{\beta\nu} \hat{\epsilon}^{\alpha\tau} . \end{aligned} \quad (19)$$

Notice that this set of equations cannot be obtained from (7) by simply splitting the field φ^i as (X^a, Y^A) . The second equation of this set has the unique solution for the inverse of $R_{\mu\nu}^{ab}$, namely $\tilde{R}_{ab}^{\mu\nu}$, given by

$$\tilde{R}_{ab}^{\mu\nu} = - \left[N_{ab}^{\mu\nu}(Y) + \hat{\epsilon}^{\mu\nu} \eta_{cd} f_{ab}^c X^d \right] \quad (20)$$

The general solution of the remaining last two equations is provided by

$$\begin{aligned} R_{\mu\nu}^{aA} &= R_{\mu\alpha}^{ab} W_{\nu b}^{\alpha A}(Y) \\ R_{AB}^{\mu\nu} &= L_{AE} L_{BD} R_{\tau\rho}^{ab} W_{\sigma a}^{\tau E} W_{\beta b}^{\rho D} \hat{\epsilon}^{\mu\sigma} \hat{\epsilon}^{\nu\beta} + T_{AB}^{\mu\nu}(Y) \end{aligned} \quad (21)$$

with $N_{ab}^{\mu\nu}$, $W_{\nu a}^{\mu A}$ and $T_{AB}^{\mu\nu}$ any arbitrary functions which depend on the field Y^A only.

Subject to these conditions, the variation with respect to X^a of our action leads to the equations of motion $\hat{\epsilon}^{\mu\nu} F_{\mu\nu}^a = 0$, where $F_{\mu\nu}^a$ is constructed from A_μ^a in (16).

Similarly, by extracting $\partial_\mu X^a$ from (16) and substituting in (15), we find (up to a total derivative)

$$\begin{aligned} S(Y, X, A) &= \int d^2x \sqrt{\gamma} \left[T_{AB}^{\mu\nu}(Y) \partial_\mu Y^A \partial_\nu Y^B + N_{ab}^{\mu\nu}(Y) A_\mu^a A_\nu^b \right. \\ &\quad \left. + 2W_{\alpha a}^{\mu A}(Y) L_{AB} \hat{\epsilon}^{\alpha\nu} A_\mu^a \partial_\nu Y^B + \hat{\epsilon}^{\mu\nu} \eta_{ab} X^a F_{\mu\nu}^b \right]. \end{aligned} \quad (22)$$

Again, the equations of motion for A_μ^a , if considered as an independent field, are precisely those in (16). The symmetry $\delta X^a = f_{bc}^a \xi^b F_{\mu\nu}^c \hat{\epsilon}^{\mu\nu}$ is transparent in this case.

The same procedure can be applied here to find the non-Abelian dual theory. Consider now the action

$$\begin{aligned} S(Y, g, A) &= \int d^2x \sqrt{\gamma} \left[T_{AB}^{\mu\nu}(Y) \partial_\mu Y^A \partial_\nu Y^B + N_{ab}^{\mu\nu}(Y) \eta^{ac} \eta^{bd} \text{tr} \left(T_c g^{-1} \partial_\mu g \right) \text{tr} \left(T_d g^{-1} \partial_\nu g \right) \right. \\ &\quad \left. + 2W_{\alpha a}^{\mu A}(Y) L_{AB} \eta^{ab} \hat{\epsilon}^{\alpha\nu} \text{tr} \left(T_b g^{-1} \partial_\mu g \right) \partial_\nu Y^B \right] \end{aligned} \quad (23)$$

which is invariant under the left symmetry $g \longrightarrow hg$. This symmetry can be gauged by the replacement $\partial_\mu g \longrightarrow D_\mu g = \partial_\mu g + A_\mu g$. The dual theory is obtained when the Lagrange multiplier term $\int d^2x \sqrt{\gamma} \hat{\epsilon}^{\mu\nu} \text{tr} (X F_{\mu\nu})$ is added. Choosing then a gauge such that $g = 1$ yields the action in (22).

Notice that in the above model (22) we have not assumed any transformation for the fields Y^A . In fact these fields could transform when $T_{AB}^{\mu\nu}$, $N_{ab}^{\mu\nu}$ and $W_{\nu a}^{\mu A}$ are restricted to satisfy certain conditions as shown below. It is found that the theory in (22), when A_μ^a is treated as an independent field, has the infinitesimal local gauge symmetry

$$\begin{aligned} \delta Y^A &= \lambda^a K_a^A(Y) \\ \delta A_\mu^a &= -\partial_\mu \lambda^a + f_{bc}^a \lambda^b A_\mu^c \end{aligned} \quad (24)$$

provided that the two quantities $T_{AB}^{\mu\nu}$ and K_a^A satisfy

$$\begin{aligned} \partial_E T_{AB}^{\mu\nu} K_a^E + T_{EB}^{\mu\nu} \partial_A K_a^E + T_{AE}^{\mu\nu} \partial_B K_a^E &= \hat{\epsilon}^{\mu\nu} (\partial_A V_{Ba} - \partial_B V_{Aa}) \\ K_a^A \partial_A K_b^B - K_b^A \partial_A K_a^B &= -f_{ab}^c K_c^B. \end{aligned} \quad (25)$$

The second equation merely expresses the fact that the differential operators $K_a = -K_a^A \frac{\partial}{\partial Y^A}$ form a representation of the Lie algebra defined by η_{ab} and f_{ab}^c . The first equation defines the new quantity V_{Aa} which is required to satisfy

$$\begin{aligned} \partial_D V_{Ab} K_c^D + \partial_A V_{Dc} K_b^D - \partial_D V_{Ac} K_b^D + V_{Db} \partial_A K_c^D &= -f_{cb}^d V_{Ad} \\ V_{Aa} K_b^A + V_{Ab} K_a^A &= 0 \quad . \end{aligned} \quad (26)$$

The remaining two quantities $N_{ab}^{\mu\nu}$ and $W_{\nu a}^{\mu A}$ are then given by

$$\begin{aligned} N_{ab}^{\mu\nu} &= T_{AB}^{\mu\nu} K_a^A K_b^B + \tilde{\epsilon}^{\mu\nu} V_{Ab} K_a^A \\ W_{\nu a}^{\mu A} &= -L^{AE} \left(\hat{\epsilon}_{\alpha\nu} T_{EB}^{\alpha\mu} K_a^B + \delta_\nu^\mu V_{Ea} \right) \quad . \end{aligned} \quad (27)$$

By writing $T_{AB}^{\mu\nu} = \gamma^{\mu\nu} G_{AB}(Y) + \hat{\epsilon}^{\mu\nu} B_{AB}(Y)$, the equations (24)–(27) are precisely the equations needed to gauge the isometries of a general sigma model with metric G_{AB} and antisymmetric tensor B_{AB} [24, 25]. Hence the non-linear sigma model obtained through a non-Abelian duality procedure is a particular case of this general construction.

4. Conclusions

It is worth mentioning that a symmetry similar to the one identified for the sigma model exists in Chern-Simons theory. To see this, consider a Chern-Simons theory for some gauge group \mathcal{G}

$$I(A) = \int d^3x \epsilon^{ijk} \left[\text{tr}(A_i F_{jk}) - \frac{2}{3} \text{tr}(A_i A_j A_k) \right] \quad (28)$$

where $i, j, \dots = 1, 2, 3$ and $\epsilon^{123} = 1$. Let also $\mu, \nu, \dots = 1, 2$ and $\epsilon^{\mu\nu}$ the corresponding alternating tensor, with $\epsilon^{12} = 1$. By splitting the three-dimensional indices, the Chern-Simons action can be written as

$$I(A) = 2 \int d^3x \epsilon^{\mu\nu} [\text{tr}(A_3 F_{\mu\nu}) - \text{tr}(A_\mu \partial_3 A_\nu) - \text{tr}(\partial_\mu (A_\nu A_3))] \quad (29)$$

It is then clear, if we drop the total divergence term, that the Chern-Simons theory has a further symmetry given by

$$A_3 \longrightarrow A_3 + \epsilon^{\mu\nu} [\xi, F_{\mu\nu}] \quad , \quad (30)$$

where ξ is a local Lie algebra-valued function. This symmetry is of the form (6). Furthermore, varying the Chern-Simons action with respect to A_3 leads to an equation similar to (9). This hints at a deep connection between Chern-Simons theory and the class of sigma models we identified as having the new symmetry. We speculate that a non trivial compactification to two dimensions of the Chern-Simons theory would lead to our sigma models.

As mentioned earlier, the new symmetry vanishes on-shell. Therefore, at the classical level this symmetry has no effects. We expect, however, that this symmetry would play a crucial role at the quantum level. We will report elsewhere on the work in progress regarding the quantisation of these models. The methods designed for the quantisation of Chern-Simons theories are essential to this investigation [26].

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References

- [1] E. Witten, Commun. Math. Phys. **92** (1984) 455.
- [2] E. Brateen, T. L. Curtright and C. K. Zackos, Nucl. Phys. **B260** (1985) 630;
T. L. Curtright and C. K. Zackos, Phys. Rev. Lett. **53** (1984) 1799.
- [3] D. Bernard, Commun. Math. Phys. **137** (1991) 191.
- [4] J. Balog, P. Forgács, Z. Horváth and L. Palla, Phys. Lett. **324** (1994) 403.
- [5] C. Ahn, D. Bernard and A. Le Clair, Nucl. Phys. **346** (1990) 409.
- [6] S. Rajeev, Phys. Lett. **217** (1989) 123.
- [7] J. M. Evans and T. J. Hollowood, Nucl. Phys. **B438(FS)** (1995) 469.
- [8] N. Mohammadi, *On Non-Abelian Duality in Sigma Models*, SHEP 95/40, hep-th/9512126, Phys. Lett. **B** (in press).
- [9] N. Mohammadi, *Classical Duality in Gauge Theories*, SHEP 95/23, hep-th/9507040.
- [10] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. **244** (1994) 77.
- [11] E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, *An Introduction to T-Duality in String Theory*, hep-th/9410237;
E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, Nucl. Phys. **424** (1994) 155.
- [12] X. de la Ossa and F. Quevedo, Nucl. Phys. **B403** (1993) 377.
- [13] A. Giveon and M. Roček, Nucl. Phys. **B421** (1994) 173.
- [14] M. Gasperini, R. Ricci and G. Veneziano, Phys. Lett. **B319** (1993) 438.

- [15] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *Remarks on Non-abelian duality*, hep-th/9409011.
- [16] O. Alvarez, *Classical Geometry and Target Space Duality*, hep-th/9511024.
- [17] E. Tyurin, Phys. Lett. **B348** (1995) 386.
- [18] S. F. Hewson, *The Non-Abelian Target Space Duals of Taub-NUT Space*, hep-th/9510092.
- [19] K. Sfetsos, Phys. Rev. **D50** (1994) 2784.
- [20] M. Roček and E. Verlinde, Nucl. Phys. **373** (1992) 630.
- [21] B. E. Fridling and A. Jevicki, Phys. Lett. **B134** (1984) 70.
- [22] E. S. Fradkin and A. A. Tseytlin, Ann. Phys. **B162** (1985) 31.
- [23] T. Curtright and C. Zachos, *Canonical Nonabelian Dual Transformations in Supersymmetric Field Theories*, hep-th/95021126;
T. Curtright, T. Uemastu and C. Zackos, *Geometry and Duality in Supersymmetric σ -Models*, hep-th/9601096;
T. Curtright and C. Zackos, Phys. Rev. **D49** (1994) 5408.
- [24] I. Jack, D. R. T. Jones, N. Mohammedi and H. Osborn, Nucl. Phys. **332** (1990) 359.
- [25] C. M. Hull and B. Spence, Phys. Lett. **B232** (1989) 204.
- [26] E. Witten, Commun. Math. Phys. **117** (1988) 353; *ibid* **118** (1988) 411; *ibid* **121** (1989) 351.